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Isomonodromic deformations with an irregular singularity and hyperelliptic curves

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Abstract

In this paper, we extend the result of Kitaev and Korotkin (1998 *Int. Math. Res. Notices* **17** 877–905) to the case where a monodromy-preserving deformation has an irregular singularity. For the monodromy-preserving deformation, we obtain the τ-function whose deformation parameters are the positions of regular singularities and the parameter t of an irregular singularity. Furthermore, the τ-function is expressed by the hyperelliptic Θ function moving the argument $\mathbf{z}$ and the period $\mathbf{B}$, where t and the positions of regular singularities move z and $\mathbf{B}$, respectively.

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1. Introduction

In this paper, we extend the result of Kitaev and Korotkin [6] to the case where a monodromy-preserving deformation has an irregular singularity. For the monodromy-preserving deformation, we obtain the τ-function represented by the hyperelliptic Θ function moving both the argument $\mathbf{z}$ and the period $\mathbf{B}$. In [3], we constructed the τ-function by the elliptic Θ function moving the argument z and the period Ω.

Miwa, Jimbo and Ueno [4] extended the work of Schlesinger [10] and established a general theory of monodromy-preserving deformation for a first-order matrix system of ordinary linear differential equations:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A(x)Y, \quad A(x) = \sum_{\nu=1}^{n}\sum_{k=0}^{r_\nu} \frac{A_{\nu,-k}}{(x-a_\nu)^{k+1}} - \sum_{k=1}^{r_\infty} A_{\infty,-k}x^{k-1}, \tag{1.1}$$

having regular or irregular singularities of arbitrary rank.

The monodromy data to be preserved are

(i) Stokes multipliers $S_j^{(\nu)}$ $(j = 1, \ldots, 2r_\nu)$,
(ii) connection matrices $C^{(\nu)}$ and
(iii) 'exponents of formal monodromy' $T_0^{(\nu)}$.

Miwa *et al* found a deformation equation as a necessary and sufficient condition for the monodromy data to be independent of deformation parameters and defined the τ-function for the deformation equation.

Let us explain the relationship between the τ-function and the Θ function. Miwa and Jimbo [5] constructed a monodromy-preserving deformation with irregular singularities and expressed the τ-function with the Θ function by moving its argument $\mathbf{z}$, which t, the parameters of the irregular singularities, move.

Kitaev and Korotkin [6] constructed a monodromy-preserving deformation of 2×2 Fuchsian systems, whose deformation parameters are the positions of $2g + 2$ regular singularities. Its τ-function is expressed by the hyperelliptic Θ function moving the period $\mathbf{B}$, which the positions of regular singularities move. When $g = 1$, the τ-function is equivalent to Picard's solution of the sixth Painlevé equation by the Bäcklund transformation. The aim of this paper is to unify Miwa and Jimbo's result and Kitaev and Korotkin's result in the hyperelliptic case. We note that Deift *et al* [1] also constructed the isomonodromic deformations of 2×2 Fuchsian systems in terms of the hyperelliptic Θ functions.

This paper is organized as follows. In section 1, we explain the basic properties of the Θ function, the prime-form and the canonical bi-meromorphic differential.

In section 2, we study an ordinary differential equation which is given by

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \left(\sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1} \right) \Psi(\lambda), \tag{1.2}$$

whose deformation parameters are $\lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$ and diagonal elements of B_{-1}. Then, following Miwa *et al* [4], we introduce the τ-function.

In section 3, we solve a class of Riemann–Hilbert (inverse monodromy) problem for special parameters. Furthermore, we prove that the solution $\Psi(\lambda)$ has the following special monodromy data:

(i) Stokes multipliers around $\lambda = \infty$ $S_1^\infty = S_2^\infty = 1$,
(ii) connection matrices around $\lambda = \lambda_j$ C_j $(1 \leqslant j \leqslant 2g + 2)$,
(iii) the exponents of formal monodromy $T_{\infty,0} = 0$, $T_{j,0} = \mathrm{diag}\left(-\frac{1}{4}, \frac{1}{4}\right)$ $(1 \leqslant j \leqslant 2g + 2)$.

In section 4, we calculate the monodromy-preserving deformation which the solution $\Psi(\lambda)$ satisfies and compute the coefficients, B_{-1}, A_j $(1 \leqslant j \leqslant 2g + 2)$.

In section 5, we find the τ-function. Section 5 consists of three subsections. Subsection 5.1 is devoted to H_t, the Hamiltonian at an irregular singular point $\lambda = \infty$. Subsection 5.2 is devoted to Fay's identities and Rauch's variational formulas. Subsection 5.3 is devoted to H_j $(1 \leqslant j \leqslant 2g + 2)$, the Hamiltonians on the deformation parameters λ_j $(1 \leqslant j \leqslant 2g + 2)$. From H_t and H_j $(1 \leqslant j \leqslant 2g + 2)$, we compute the τ-function and prove our main theorem.

In order to state our main theorem, we define hyperelliptic curves and the Θ function and introduce the prime-form and the canonical bi-meromorphic differential, following Fay [2]. The hyperelliptic curves $\mathcal{L}$ are

$$\omega^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j),$$

whose branch points $\lambda_j \in \mathbb{C}$ $(1 \leqslant j \leqslant 2g + 2)$ are distinct and canonical homological basis $\{a_k, b_k\}_{1 \leqslant k \leqslant g}$ are chosen according to figure 1.

The basic holomorphic one-forms are expressed by

$$\mathrm{d}U_k^0 = \frac{\lambda^{k-1}\mathrm{d}\lambda}{\omega}, \qquad 1 \leqslant k \leqslant g.$$

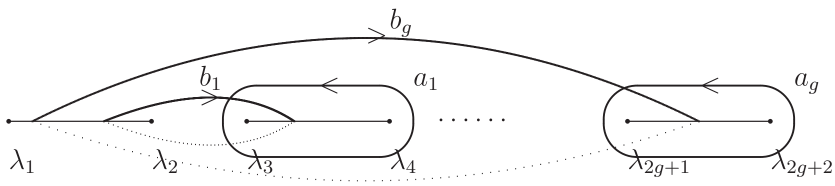

Figure 1. Branch cuts and canonical basis of cycles on the hyperelliptic curves, $\mathcal{L}$. Continuous and dashed paths lie on the first and second sheet of $\mathcal{L}$, respectively.

Then, the $g \times g$ matrices of a- and b-periods are given by

$$\mathcal{A}_{kl} = \oint_{a_l} \mathrm{d}U_k^0, \qquad \mathcal{B}_{kl} = \oint_{b_l} \mathrm{d}U_k^0, \qquad 1 \leqslant k, l \leqslant g.$$

Thus, the normalized holomorphic one-forms are defined by

$$\mathrm{d}U_k = \frac{1}{\omega} \sum_{l=1}^{g} (\mathcal{A}^{-1})_{kl} \lambda^{l-1} \mathrm{d}\lambda \qquad 1 \leqslant k \leqslant g,$$

which satisfy

$$\oint_{a_l} \mathrm{d}U_k = \delta_{kl}.$$

Furthermore, from $\mathcal{A}$ and $\mathcal{B}$, we can construct

$$\mathbf{B} = \mathcal{A}^{-1}\mathcal{B}.$$

Following Mumford [8], we define the Θ function with characteristic $[\mathbf{p}, \mathbf{q}]$ $(\mathbf{p}, \mathbf{q} \in \mathbb{C}^g)$ by

$$\begin{aligned}\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}|\mathbf{B}) &= \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}) \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\{\pi \mathrm{i} \langle \mathbf{B}(\mathbf{m} + \mathbf{p}), \mathbf{m} + \mathbf{p} \rangle + 2\pi \mathrm{i} \langle \mathbf{z} + \mathbf{q}, \mathbf{m} + \mathbf{p} \rangle\},\end{aligned}$$

where $\mathbf{z} \in \mathbb{C}^g$ and the sum extends over all integer vectors in $\mathbb{C}^g$.

Following Fay [2], we introduce the prime-form $E(P,Q)$ and the canonical bi-meromorphic differential $W(P,Q)$. The prime-form is defined by

$$E(P, Q) = \frac{\Theta[\mathbf{p}^*, \mathbf{q}^*](U(P) - U(Q))}{h_*(P) h_*(Q)}$$

$$(h_*(P))^2 = \sum_{k=1}^{g} \frac{\partial \Theta[\mathbf{p}^*, \mathbf{q}^*]}{\partial z_k} (0|\mathbf{B})\, \mathrm{d}U_k(P) \qquad \text{for} \quad P, Q \in \mathcal{L},$$

where $[\mathbf{p}^*, \mathbf{q}^*]$ is an arbitrary odd non-singular half-integer characteristic. The canonical bi-meromorphic differential $W(P,Q)$ is given by

$$W(P, Q) = \mathrm{d}x_P\, \mathrm{d}x_Q \log E(P, Q),$$

where $\mathrm{d}x_P$, $\mathrm{d}x_Q$ are the exterior differentiations with respect to the local parameter of P, Q, respectively. When $P = \infty^1$ and $Q = \infty^2$, we define the local coordinates of P, Q by

$$x_{\infty^1} = \frac{1}{\lambda}, \qquad x_{\infty^2} = \frac{1}{\lambda},$$

and have

$$W(\infty^1, \infty^2) = \left(\frac{\partial^2}{\partial x_{\infty^1} \partial x_{\infty^2}} \log E(P, Q) \Big|_{P=\infty^1, Q=\infty^2} \right) \mathrm{d}x_{\infty^1} \, \mathrm{d}x_{\infty^2}.$$

According to Fay [2], the projective connection $S(Q)$ is given by the equation

$$W(P, Q) = \left(\frac{1}{(x_P - x_Q)^2} + \frac{1}{6} S(Q) + O(x_P - x_Q) \right) \mathrm{d}x_P \, \mathrm{d}x_Q,$$

where P and Q have local coordinates x_P, x_Q in a neighborhood of $Q \in \mathcal{L}$. When $Q = \infty^1$, we define the local coordinate of Q by

$$x_P = x_Q = \frac{1}{\lambda} \qquad \text{if} \quad P \longrightarrow Q = \infty^1.$$

Our main theorem is as follows.

Theorem 1.1. *For the monodromy-preserving deformation (5.1), the τ-function is*

$$\tau(\lambda_1, \ldots, \lambda_{2g+2}, t) = \Theta[\mathbf{p}, \mathbf{q}] \, (\mathbf{v}(t)|\mathbf{B}) \, (\det \mathcal{A})^{-\frac{1}{2}} \prod_{1 \leqslant j < k \leqslant 2g+2} (\lambda_j - \lambda_k)^{-\frac{1}{8}}$$
$$\times \exp \left\{ \frac{t^2}{4} \left(\frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{\mathrm{d}x_{\infty^1} \, \mathrm{d}x_{\infty^2}} \right) \right\},$$

where

$$\mathbf{v}(t) = t \times \left(\frac{\mathrm{d}U_1}{\mathrm{d}x_{\infty^1}}(\infty^1), \frac{\mathrm{d}U_2}{\mathrm{d}x_{\infty^1}}(\infty^1), \ldots, \frac{\mathrm{d}U_g}{\mathrm{d}x_{\infty^1}}(\infty^1) \right)$$

and $x_{\infty^1} = \frac{1}{\lambda}$, *which is a local coordinate of* ∞^1.

Remark. By setting $t = 0$, we obtain Kitaev and Korotkin's τ-function in [6]. We had a branch point at ∞ in [3], but we do not have it in this paper.

2. Hyperelliptic curves and the Θ function

In this section, we explain the detailed properties of the Θ function, the prime-form $E(P,Q)$ and the canonical bi-meromorphic differential $W(P,Q)$.

The Θ function possesses the following periodicity properties:

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} + \mathbf{e}_k|\mathbf{B}) = \exp\{2\pi \mathrm{i} p_k\} \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}|\mathbf{B}) \tag{2.1}$$

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} + \mathbf{B}\mathbf{e}_k|\mathbf{B}) = \exp\{-2\pi \mathrm{i} q_k - \pi \mathrm{i} \mathbf{B}_{kk} - 2\pi \mathrm{i} z_k\} \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}|\mathbf{B}), \tag{2.2}$$

where

$$\mathbf{e}_k = {}^t(0, \ldots, \overset{k\text{th}}{1}, \ldots, 0).$$

We define the Abel map $U(P) \in \mathbb{C}^g$ $P \in \mathcal{L}$ by

$$U(P) = {}^t \left(U_1(P), U_2(P), \ldots, U_g(P) \right)$$
$$U_k(P) = \int_{\lambda_1}^{P} \mathrm{d}U_k \quad (1 \leqslant k \leqslant g).$$

For the canonical homological basis $\{a_k, b_k\}_{1 \leqslant k \leqslant g}$ and the base point λ_1, the Riemann constants $\mathbf{K} \in \mathbb{C}^g$ are as follows:

$$\mathbf{K} = \tfrac{1}{2}\mathbf{B}(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_g) + \tfrac{1}{2}(\mathbf{e}_1 + 2\mathbf{e}_2 + \cdots + g\mathbf{e}_g).$$

A characteristic $[\mathbf{p}, \mathbf{q}]$ is a $g \times 2$ matrix of complex numbers which is given by

$$[\mathbf{p}, \mathbf{q}] = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \\ \vdots & \vdots \\ p_g & q_g \end{bmatrix},$$

where

$$\mathbf{p} = {}^t(p_1, p_2, \ldots, p_g), \qquad \mathbf{q} = {}^t(q_1, q_2, \ldots, q_g).$$

We consider p_i, q_i $(1 \leqslant \mathrm{i} \leqslant g)$ as elements of $\mathbb{C}^g/\mathbb{Z}^g$. If all the components are half-integers, $[\mathbf{p}, \mathbf{q}]$ is called a half-integer characteristic. A half-integer characteristic is in one-to-one correspondence with a half-period $\mathbf{Bp} + \mathbf{q}$. If the scalar product $4\langle \mathbf{p}, \mathbf{q}\rangle$ is odd, then the characteristic is called odd and the related Θ function is odd with respect to its argument $\mathbf{z}$. If this scalar product is even, then the characteristic is called even and the related Θ function is even with respect to its argument $\mathbf{z}$.

The odd characteristics which are important for us in the following correspond to any subset

$$S = \{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{g-1}}\},$$

whose components are arbitrary distinct branch points. The odd half-period corresponding to the subset S is expressed by

$$\mathbf{Bp}^S + \mathbf{q}^S = U(\lambda_{i_1}) + U(\lambda_{i_2}) + \cdots + U(\lambda_{i_{g-1}}) - \mathbf{K}.$$

Analogously, we shall be interested in the even half-period corresponding to the subset

$$T = \{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{g+1}}\},$$

which consists of arbitrary $g + 1$ branch points. The even half-period is given by

$$\mathbf{Bp}^T + \mathbf{q}^T = U(\lambda_{i_1}) + U(\lambda_{i_2}) + \cdots + U(\lambda_{i_{g+1}}) - \mathbf{K}.$$

We fix the choice of T and define

$$\{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_{g+1}}\} := \{\lambda_1, \lambda_2, \ldots, \lambda_{2g+2}\} \setminus T.$$

We explain the detailed property of $E(P,Q)$. On page 13–14 of Fay [2], we find

$$E(P, Q) = \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](U(P) - U(Q))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](0) m_T(P, Q)},$$

$$m_T(P, Q) = \frac{\omega(Q)\prod_{k=1}^{g+1}(\lambda(P) - \lambda_{i_k}) + \omega(P)\prod_{k=1}^{g+1}(\lambda(Q) - \lambda_{i_k})}{2(\lambda(Q) - \lambda(P))} \times \left[\frac{\mathrm{d}\lambda(P)\,\mathrm{d}\lambda(Q)}{\omega(P)\omega(Q)\prod_{k=1}^{g+1}(\lambda(P) - \lambda_{i_k})(\lambda(Q) - \lambda_{i_k})}\right]^{\frac{1}{2}},$$

because $\mathcal{L}$ is hyperelliptic.

3. The Schlesinger system

We study an ordinary differential equation which is given by

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \left(\sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1}\right)\Psi(\lambda), \tag{3.1}$$

where $A_1, A_2, \ldots, A_{2g+2}, B_{-1} \in sl(2, \mathbb{C})$ are independent of λ.

The monodromy data of (3.1) are as follows:

(i) Stokes multipliers around $\lambda = \infty$ $S_1^\infty = S_2^\infty$;
(ii) connection matrices around λ_j C_j $(j = 1, 2, \ldots, 2g+2)$ and
(iii) the exponents of formal monodromy $T_{\infty,0}, T_{j,0}$ $(j = 1, 2, \ldots, 2g+2)$.

In the next section, we obtain a convergent series around $\lambda = \infty, \lambda_j$ $(j = 1, 2, \ldots, 2g+2)$ which are expressed by

$$\Psi(\lambda) = \left(1 + O\left(\frac{1}{\lambda}\right)\right) \exp T^\infty(\lambda) = \hat{\Psi}^\infty(\lambda) \exp T^\infty(\lambda), \tag{3.2}$$

$$\Psi(\lambda) = G_j(1 + O(\lambda - \lambda_j)) \exp T_j(\lambda) = G_j \hat{\Psi}_j(\lambda) \exp T_{j,0}(\lambda) \quad (j = 1, 2, \ldots, 2g+2), \tag{3.3}$$

where

$$T^\infty(\lambda) = \begin{pmatrix} -\frac{t}{2} & \\ & \frac{t}{2} \end{pmatrix} \lambda + T_{\infty,0} \log\left(\frac{1}{\lambda}\right), \tag{3.4}$$

$$T_j(\lambda) = T_{j,0} \log(\lambda - \lambda_j) \qquad (j = 1, 2, \ldots, 2g+2). \tag{3.5}$$

For the deformation parameters $t, \lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$, the closed one-form is defined by

$$\Omega = \omega_\infty + \omega_{\lambda_1} + \omega_{\lambda_2} + \cdots + \omega_{\lambda_{2g+2}} \tag{3.6}$$

$$= H_t\, \mathrm{d}t + H_1\, \mathrm{d}\lambda_1 + H_2\, \mathrm{d}\lambda_2 + \cdots + H_{2g+2}\, \mathrm{d}\lambda_{2g+2}, \tag{3.7}$$

where

$$\omega_\infty = -\mathrm{Res}_{\lambda=\infty} \mathrm{tr} \hat{\Psi}^\infty(\lambda)^{-1} \frac{\partial \hat{\Psi}^\lambda}{\partial \lambda}(\lambda)\, \mathrm{d}T^\infty(\lambda), \tag{3.8}$$

$$\omega_{\lambda_j} = -\mathrm{Res}_{\lambda=\lambda_j} \mathrm{tr} \hat{\Psi}_j(\lambda)^{-1} \frac{\partial \hat{\Psi}_j}{\partial \lambda}(\lambda)\, \mathrm{d}T_j(\lambda) \qquad (j = 1, 2, \ldots, 2g+2), \tag{3.9}$$

and d is the exterior differentiation with respect to the deformation parameters $t, \lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$. Especially, we can write

$$\omega_{\lambda_j} = \left[\mathrm{Res}_{\lambda=\lambda_j} \frac{1}{2} \mathrm{tr} \left(\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} \Psi^{-1}\right)^2\right] \mathrm{d}\lambda_j. \tag{3.10}$$

Then, from the closed one-form Ω, the τ-function is defined by

$$\Omega := d \log \tau(\lambda_1, \lambda_2, \ldots, \lambda_{2g+2}, t). \tag{3.11}$$

4. The Riemann–Hilbert problem for special parameters

In this section, we concretely construct a 2×2 matrix valued function $\Psi(\lambda)$, whose monodromy data, (i) Stokes multipliers, (ii) connection matrices, (iii) exponents of formal monodromy, are independent of the deformation parameters $t, \lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$.

The involution of $\mathcal{L}$ is defined by

$$* : (\lambda, \omega) \longrightarrow (\lambda, -\omega).$$

Then, from the Θ function and $W(P,Q)$, we define the 2×2 matrix valued function $\Phi(P)$ by

$$\Phi(P) = \begin{pmatrix} \varphi(P) & \varphi(P^*) \\ \psi(P) & \psi(P^*) \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned} \varphi(P) &= \Theta[\mathbf{p},\mathbf{q}](U(P)+U(P_\varphi)+\mathbf{v}(t))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(P)-U(P_\varphi)) \\ &\quad\times \exp -\frac{t}{2}\{-\int_{\lambda_1}^{P} W(P,\infty^1) + \int_{\lambda_1}^{P} W(P,\infty^2)\}, \\ \psi(P) &= \Theta[\mathbf{p},\mathbf{q}](U(P)+U(P_\psi)+\mathbf{v}(t))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(P)-U(P_\psi)) \\ &\quad\times \exp -\frac{t}{2}\{-\int_{\lambda_1}^{P} W(P,\infty^1) + \int_{\lambda_1}^{P} W(P,\infty^2)\}, \\ \mathbf{v}(t) &= t \times \left(\frac{\mathrm{d}U_1}{\mathrm{d}x_{\infty^1}}(\infty^1), \frac{\mathrm{d}U_2}{\mathrm{d}x_{\infty^1}}(\infty^1), \ldots, \frac{\mathrm{d}U_g}{\mathrm{d}x_{\infty^1}}(\infty^1)\right), \end{aligned}$$

where P_φ, P_ψ are arbitrary points of $\mathcal{L}$ and $x_{\infty^1} = \frac{1}{\lambda}$, which is a local coordinate of ∞^1.

Proposition 4.1.

(1) The function $\Phi(P)$ is invertible outside of the branch points $\lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$.

(2) $\det \Phi(P)$ has zeros at $\lambda_j \notin S$ with the first order and has zeros at $\lambda_j \in S$ with the third order.

(3) The function $\Phi(P)$ transforms as follows with respect to the tracing along the canonical homological basis, a_k, b_k $(k = 1, 2, \ldots, g)$:

$$T_{a_k}[\Phi(P)] = \Phi(P) \begin{pmatrix} \exp\{2\pi \mathrm{i}(p_k + p_k^S)\} & \\ & \exp\{-2\pi \mathrm{i}(p_k + p_k^S)\} \end{pmatrix}, \tag{4.2}$$

$$\begin{aligned} T_{b_k}[\Phi(P)] &= \Phi(P) \begin{pmatrix} \exp\{-2\pi \mathrm{i}(q_k + q_k^S)\} & \\ & \exp\{2\pi \mathrm{i}(q_k + q_k^S)\} \end{pmatrix} \\ &\quad \times \exp\{-2\pi \mathrm{i}\mathbf{B}_{kk} - 4\pi \mathrm{i} U_k(P)\}, \end{aligned} \tag{4.3}$$

where T_l denotes the operator of analytic continuation along the contour l.

Proof. By using the periodicity (2.1) and (2.2), we obtain

$$\begin{aligned} T_{a_k}[\varphi(P)] &= \exp\{2\pi \mathrm{i}(p_k + p_k^S)\}\varphi(P), \\ T_{b_k}[\varphi(P)] &= \exp\{-2\pi \mathrm{i}(q_k + q_k^S) - 2\pi \mathrm{i}\mathbf{B}_{kk} - 4\pi \mathrm{i} U_k(P)\}\varphi(P). \end{aligned}$$

We deduce the same transformation laws for $\psi(P)$.

The actions of the involution $*$ on $\{a_k, b_k\}$ and $\mathrm{d}U_k$ are given by

$$a_k^* = -a_k, b_k^* = -b_k, \mathrm{d}U_k(P^*) = -\mathrm{d}U_k(P) \qquad (k = 1, 2, \ldots, g),$$

respectively. Therefore, we get

$$T_{a_k}[\varphi(P^*)] = \exp\{-2\pi \mathrm{i}(p_k + p_k^S)\}\varphi(P^*), \tag{4.4}$$

$$T_{b_k}[\varphi(P^*)] = \exp\{2\pi \mathrm{i}(q_k + q_k^S) - 2\pi \mathrm{i}\mathbf{B}_{kk} - 4\pi \mathrm{i} U_k(P)\}\varphi(P^*). \tag{4.5}$$

We deduce the same transformation laws for $\psi(P^*)$. Then, we complete the proof of (3).

Equations (4.2) and (4.3) imply that for $k = 1, 2, \ldots, g$,

$$T_{a_k}[\det \Phi(P)] = \det \Phi(P),$$

$$T_{b_k}[\det \Phi(P)] = \det \Phi(P) \exp\{-4\pi \mathrm{i}\mathbf{B}_{kk} - 8\pi \mathrm{i} U_k(P)\},$$

which imply that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial \hat{\mathcal{L}}} \frac{d(\det \Phi(P))}{\det \Phi(P)} = 4g.$$

Thus, it follows that

$$3(g-1) + g + 3 = 4g,$$

because $\det \Phi(P)$ has zeros at the branch points λ_j and has zeros at $\lambda_j \in S$ of the order of at least 3. Therefore, it follows that $\det \Phi(P)$ does not vanish outside of the branch points and that $\det \Phi(P)$ has zeros at $\lambda_j \notin S$ with the first order and has zeros at $\lambda_j \in S$ with the third order, which complete the proof of (1) and (2), respectively. □

In order to normalize $\Phi(\lambda)$ near $\lambda = \infty$, we have

Lemma 4.2. *For $P, Q \in \mathcal{L}$,*

$$W(P, Q) = \left(\frac{1}{(x_P - x_Q)^2} + \frac{1}{6} S(Q) + O(x_P - x_Q)\right) \mathrm{d}x_P\, \mathrm{d}x_Q, \tag{4.6}$$

$$\begin{aligned} \frac{1}{6} S(Q) = {} & \frac{1}{6}\{\lambda, x_Q\}(Q) + \frac{1}{16}\left(\frac{\mathrm{d}}{\mathrm{d}x_Q} \log \prod_{k=1}^{g+1} \frac{\lambda - \lambda_{i_k}}{\lambda - \lambda_{j_k}}(Q)\right)^2 \\ & - \sum_{k,l=1}^{g} \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{\mathrm{d}U_k}{\mathrm{d}x_Q}(Q) \frac{\mathrm{d}U_l}{\mathrm{d}x_Q}(Q), \end{aligned} \tag{4.7}$$

where x_P, x_Q are local coordinates of $P, Q \in \mathcal{L}$ and

$$\{\lambda, x\} = \frac{\lambda'''}{\lambda'} - \frac{3}{2}\left(\frac{\lambda''}{\lambda'}\right)^2.$$

Proof. See p 20 in [2]. □

In lemma 4.2, we set

$$x_P = x_Q = \frac{1}{\lambda}, \qquad Q = \infty^1,$$

and take the limit $P \longrightarrow \infty^1$. Furthermore, we define the constant terms $c_{\infty^1}, c_{\infty^2}$ by

$$\int_{\lambda_1}^{P} W(P, \infty^1) = -\lambda + c_{\infty^1} + \frac{1}{6} S(\infty^1)\lambda^{-1} + \cdots \text{near } \lambda = \infty^1,$$

$$\int_{\lambda_1}^{P} W(P, \infty^2) = c_{\infty^2} + \frac{W(\infty^1, \infty^2)}{\mathrm{d}x_{\infty^1} \mathrm{d}x_{\infty^2}} \lambda^{-1} + \cdots \text{near } \lambda = \infty^1.$$

Therefore, $\Phi(\lambda)$ can be developed near $\lambda = \infty$ as

$$\Phi(\lambda) = \left(G^\infty + O\left(\frac{1}{\lambda}\right)\right) \exp\begin{pmatrix} -\frac{t}{2}\lambda & \\ & \frac{t}{2}\lambda \end{pmatrix},$$

where G^∞ is a 2×2 matrix whose matrix elements are given by

$$(G^\infty)_{11} = \Theta[\mathbf{p},\mathbf{q}](U(\infty^1) + U(P_\varphi) + \mathbf{v}(t))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(\infty^1) - U(P_\varphi)) \times \exp\left\{\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\},$$

$$(G^\infty)_{21} = \Theta[\mathbf{p},\mathbf{q}](U(\infty^1) + U(P_\psi) + \mathbf{v}(t))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(\infty^1) - U(P_\psi)) \times \exp\left\{\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\},$$

$$(G^\infty)_{12} = \Theta[\mathbf{p},\mathbf{q}](U(\infty^2) + U(P_\varphi) + \mathbf{v}(t))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(\infty^2) - U(P_\varphi)) \times \exp\left\{-\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\},$$

$$(G^\infty)_{22} = \Theta[\mathbf{p},\mathbf{q}](U(\infty^2) + U(P_\psi) + \mathbf{v}(t))\Theta[\mathbf{p}^S,\mathbf{q}^S](U(\infty^2) - U(P_\psi)) \times \exp\left\{-\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\}.$$

Proposition 4.1 shows that

$$\det G^\infty = \det \Phi(\infty) \neq 0.$$

We define a matrix valued function $\Psi(\lambda)$ by

$$\Psi(P) = \frac{\sqrt{\det \Phi(\infty)}}{\sqrt{\det \Phi(P)}}(G^\infty)^{-1}\Phi(P). \tag{4.8}$$

The expansions of $\Psi(\lambda)$ near $\lambda = \infty$ are as follows:

Lemma 4.3.

$$\Psi(\lambda) = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + O\left(\frac{1}{\lambda}\right)\right)\exp T^\infty(\lambda),$$

$$T^\infty(\lambda) = T^\infty_{-1}\lambda,\ T^\infty_{-1} = \begin{pmatrix}-\frac{t}{2} & \\ & \frac{t}{2}\end{pmatrix},$$

where the Taylor series of $\Psi(\lambda)\exp\{-T^\infty(\lambda)\}$ *is convergent. Especially, if* $P_\varphi = \infty^1$ *and* $P_\psi = \infty^2$,

$$\Psi(\lambda) = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \Psi^\infty_{-1}\lambda^{-1} + \cdots\right)\exp T^\infty(\lambda),$$

$$\left(\Psi^\infty_{-1}\right)_{11} = \sum_{k=1}^{g}\frac{\partial}{\partial z_k}\log\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))\frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^1}}(\infty^1) + \frac{t}{2}\left(\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\mathrm{d}x_{\infty^2}}\right),$$

$$\left(\Psi^\infty_{-1}\right)_{21} = \frac{\mathrm{i}}{E(\infty^2,\infty^1)}\frac{\Theta[\mathbf{p},\mathbf{q}](2U(\infty^1) + \mathbf{v}(t))}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\exp\{t\,(c_{\infty^1} - c_{\infty^2})\},$$

$$\left(\Psi^\infty_{-1}\right)_{12} = \frac{\mathrm{i}}{E(\infty^1,\infty^2)}\frac{\Theta[\mathbf{p},\mathbf{q}](2U(\infty^2) + \mathbf{v}(t))}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\exp\{-t\,(c_{\infty^1} - c_{\infty^2})\},$$

$$\left(\Psi^\infty_{-1}\right)_{22} = \sum_{k=1}^{g}\frac{\partial}{\partial z_k}\log\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))\frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^2}}(\infty^2) - \frac{t}{2}\left(\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\mathrm{d}x_{\infty^2}}\right).$$

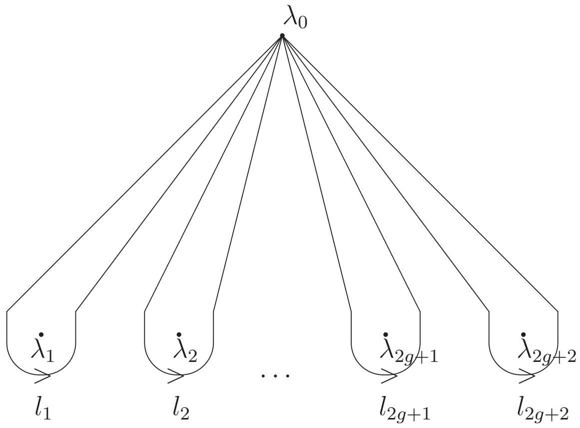

Figure 2. Generators of $\pi_1\left(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_{2g+2}\}, \lambda_0\right)$.

In the following theorem, we determine the monodromy matrices and the Stokes matrices of $\Psi(\lambda)$.

Theorem 4.4. *For $j = 1, 2, \ldots, 2g+2$, the monodromy matrix M_j of $\Psi(\lambda)$ corresponding to the contour l_j, defined in figure 2 is given by*

$$M_j = \begin{pmatrix} 0 & m_j \\ -m_j^{-1} & 0 \end{pmatrix}, \tag{4.9}$$

where

$$m_1 = -\mathrm{i},\ m_2 = \mathrm{i}(-1)^{g+1} \exp\left\{-2\pi\mathrm{i}\sum_{l=1}^{g} p_l\right\},$$

$$m_{2k+1} = \mathrm{i}(-1)^{g} \exp\left\{2\pi\mathrm{i}q_k - 2\pi\mathrm{i}\sum_{l=k}^{g} p_l\right\},$$

$$m_{2k+2} = \mathrm{i}(-1)^{g+1} \exp\left\{2\pi\mathrm{i}q_k - 2\pi\mathrm{i}\sum_{l=k+1}^{g} p_l\right\}$$

for $k = 1, 2, \ldots, g$. The Stokes matrices are expressed by

$$S_1^\infty = S_2^\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. By the involution $*$, we get

$$\Psi(\lambda)M_1 = \Psi(\lambda)\begin{pmatrix} & \mathrm{i} \\ & \end{pmatrix}, \quad \text{or} = \Psi(\lambda)\begin{pmatrix} & -\mathrm{i} \\ -\mathrm{i} & \end{pmatrix}.$$

We define

$$M_1 = \begin{pmatrix} & m_1 \\ -m_1^{-1} & \end{pmatrix} = \begin{pmatrix} & -\mathrm{i} \\ -\mathrm{i} & \end{pmatrix}.$$

Proposition 4.1 implies that

$$T_{a_k}[\Psi(\lambda)] = \Psi(\lambda)M_{2k+2}M_{2k+1}$$
$$= \Psi(\lambda)\frac{T_{l_{2k+1}\circ l_{2k+2}}[\sqrt{\det\Phi(P)}]}{\sqrt{\det\Phi(P)}}\exp\left\{2\pi\mathrm{i}\left(p_k + p_k^S\right)\right\}\sigma_3$$

and

$$T_{-b_k+b_{k-1}}[\Psi(\lambda)] = \Psi(\lambda)M_{2k+1}M_{2k}$$
$$= \Psi(\lambda)\frac{T_{l_{2k}\circ l_{2k+1}}[\sqrt{\det\Phi(P)}]}{\sqrt{\det\Phi(P)}}\exp\{2\pi\mathrm{i}(q_k - q_{k-1} + q_k - q_{k-1})\}\sigma_3,$$

where $\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. In order to determine the monodromy matrices, we have

$$U(\lambda_1) = 0, \qquad U(\lambda_2) = \frac{1}{2}\sum_{l=1}^{g}\mathbf{e}_l,$$

$$U(\lambda_{2k+1}) = \frac{1}{2}\mathbf{B}\mathbf{e}_k + \frac{1}{2}\sum_{l=k}^{g}\mathbf{e}_l, \qquad U(\lambda_{2k+2}) = \frac{1}{2}\mathbf{B}\mathbf{e}_k + \frac{1}{2}\sum_{l=k+1}^{g}\mathbf{e}_l \qquad (k = 1, 2, \ldots, g).$$

Then, we get

$$p_k^S = \frac{1}{2}(\delta_{2k+1} + \delta_{2k+2} + 1), \qquad q_{k+1} - q_k = \frac{1}{2}(\delta_{2k+2} + \delta_{2k+3} + 1),$$

where

$$\begin{cases}\delta_j = 1 & \text{if} \quad \lambda_j \in S\\ \delta_j = 0 & \text{if} \quad \lambda_j \notin S.\end{cases}$$

The function $\sqrt{\det\Phi(P)}$ transforms with respect to the tracing along the cycles l_j $(j = 1, 2, \ldots, 2g+2)$ in the following way:

$$\begin{cases}T_{l_{2k+1}\circ l_{2k+2}}[\sqrt{\det\Phi(P)}] = \exp\{\pi\mathrm{i}(\delta_{2k+1} + \delta_{2k+2} + 1)\}\sqrt{\det\Phi(P)},\\ T_{l_{2k}\circ l_{2k+1}}[\sqrt{\det\Phi(P)}] = \exp\{\pi\mathrm{i}(\delta_{2k+2} + \delta_{2k+3} + 1)\}\sqrt{\det\Phi(P)}.\end{cases} \tag{4.10}$$

In order to prove (4.10), we have only to note that if λ_j is in S, from proposition 4.1, $\det\Phi(P)$ has a zero of order 1 at $\lambda = \lambda_j$ and that if λ_j is not in S, from proposition 4.1, $\det\Phi(P)$ has a zero of order 3 at $\lambda = \lambda_j$. Therefore, we obtain

$$\begin{cases}M_{2k+2}M_{2k+1} = \exp\{2\pi\mathrm{i}\sigma_3\}\\ M_{2k+1}M_{2k} = \exp\{2\pi\mathrm{i}(q_k - q_{k-1})\sigma_3\}.\end{cases}$$

By considering

$$M_{2g+2}M_{2g+1}\cdots M_1 = I,$$

we get the monodromy matrices.

Because $\Psi(\lambda)\exp\{-T^\infty(\lambda)\}$ can be developed near $\lambda = \infty$ as a convergent series, the Stokes multipliers are $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. □

We can describe the monodromy data of $\Psi(\lambda)$ in the following way.

Corollary 4.5. $\Psi(\lambda)$ *has the following monodromy data:*

(i) Stokes multipliers $S_1^\infty = S_2^\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

(ii) connection matrices $C_j = \frac{1}{\sqrt{2\mathrm{i}m_j}} \begin{pmatrix} 1 & \mathrm{i}m_j \\ -1 & \mathrm{i}m_j \end{pmatrix} (j = 1, 2, \ldots, 2g+2)$ *and*

(iii) exponents of formal monodromy $T_{j,0} = \mathrm{diag}(-\frac{1}{4}, \frac{1}{4})$ $(j = 1, 2, \ldots, 2g+2)$.

Specifically, the developments of $\Psi(\lambda)$ *near* $\lambda = \lambda_j$ $(j = 1, 2, \ldots, 2g+2)$ *are expressed by*

$$\Psi(\lambda) = G_j(1 + O(\lambda - \lambda_j)) \exp T_j(\lambda) C_j,$$
$$T_j(\lambda) = T_{j,0} \log(\lambda - \lambda_j).$$

Proof. (i) is clear. (ii) and (iii) can be obtained by diagonalizing the monodromy matrices M_j $(j = 1, 2, \ldots, 2g+2)$. □

This corollary means that the monodromy data of $\Psi(\lambda)$ are independent of the deformation parameters $t, \lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$.

5. Monodromy-preserving deformation

In this section, we prove that $\Psi(\lambda)$ satisfies an ordinary differential equation which is expressed by

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \left(\sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1} \right) \Psi(\lambda),$$

and concretely determine the coefficients B_{-1}, A_j $(1 \leqslant j \leqslant 2g+2)$.

We note that the monodromy matrices and the Stokes matrices are independent of the parameters $P_\varphi, P_\phi, p_k^S, q_k^S$ $(1 \leqslant k \leqslant g)$ in theorem 4.4. Thus, in this section, we set $P_\varphi = \infty^2$, $P_\phi = \infty^1$and take S_j so that λ_j is not in S_j because of the uniqueness of a solution of the Riemann–Hilbert problem. Therefore, we get

$$\Psi(\lambda) = \frac{1}{\sqrt{\det \Phi^\infty(\lambda)}} \Phi^\infty(\lambda),$$

where

$$\Psi(P) = \begin{pmatrix} \varphi_j^\infty(P) & \varphi_j^\infty(P^*) \\ \psi_j^\infty(P) & \psi_j^\infty(P^*) \end{pmatrix},$$

$$\varphi_j^\infty(P) = \frac{\Theta[\mathbf{p}, \mathbf{q}](U(P) + U(\infty^2) + \mathbf{v}(t))\Theta[\mathbf{p}^{S_j}, \mathbf{q}^{S_j}](U(P) - U(\infty^2))}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))\Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(\infty^2))}$$
$$\times \exp\left\{-\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\} \exp \Pi(P),$$

$$\psi_j^\infty(P) = \frac{\Theta[\mathbf{p}, \mathbf{q}](U(P) + U(\infty^1) + \mathbf{v}(t))\Theta[\mathbf{p}^{S_j}, \mathbf{q}^{S_j}](U(P) - U(\infty^1))}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))\Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(\infty^1))}$$
$$\times \exp\left\{\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\} \exp \Pi(P).$$

Theorem 5.1. $\Psi(\lambda)$ *satisfies the following ordinary differential equation:*

$$\frac{d\Psi}{d\lambda} = \left(\sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1}\right)\Psi(\lambda), \tag{5.1}$$

where

$$B_{-1} = \operatorname{diag}\left(\frac{t}{2}, -\frac{t}{2}\right),$$

$$A_j = -\frac{1}{4}F_j^{\infty}\sigma_3(F_j^{\infty})^{-1}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$F_j^{\infty} = \begin{pmatrix} \varphi_j^{\infty}(\lambda_j) & \frac{d}{dx_j}\varphi_j^{\infty}(\lambda_j) \\ \psi_j^{\infty}(\lambda_j) & \frac{d}{dx_j}\psi_j^{\infty}(\lambda_j) \end{pmatrix},$$

and $x_j = \sqrt{\lambda - \lambda_j}$.

Proof. From lemma 4.3, it follows that

$$\Psi_\lambda \Psi^{-1} = \begin{pmatrix} -\frac{t}{2} & \\ & \frac{t}{2} \end{pmatrix} + O(\lambda^{-1}) := -B_{-1} + O(\lambda^{-1}).$$

Near $\lambda = \lambda_j$, we have

$$\begin{cases} \varphi_j^{\infty}(P) = \varphi_j(\lambda_j) + \sqrt{\lambda - \lambda_j}\frac{d}{dx_j}\varphi_j(\lambda_j) + \cdots \\ \psi_j^{\infty}(P) = \psi_j(\lambda_j) + \sqrt{\lambda - \lambda_j}\frac{d}{dx_j}\psi_j(\lambda_j) + \cdots, \end{cases}$$

which implies that

$$\det \Phi^{\infty}(P) = -2\sqrt{\lambda - \lambda_j}\det F_j^{\infty} + O(\lambda - \lambda_j).$$

From the definition of S_j, it follows that $\det F_j^{\infty} \neq 0$.

We set

$$\Psi(\lambda) := G_j(1 + O(\lambda - \lambda_j))(\lambda - \lambda_j)^{\begin{pmatrix} -\frac{1}{4} & \\ & \frac{1}{4} \end{pmatrix}} C_j.$$

Then, we get

$$\begin{aligned} G_j = \Psi(\lambda)C_j^{-1}(\lambda - \lambda_j)^{\begin{pmatrix} \frac{1}{4} & \\ & -\frac{1}{4} \end{pmatrix}}\Bigg|_{\lambda=\lambda_j} &= (-2\det F_j^{\infty})^{-\frac{1}{2}}(1 + O(\sqrt{\lambda - \lambda_j})) \\ &\quad \times \Phi^{\infty}(P)C_j^{-1}(\lambda - \lambda_j)^{\begin{pmatrix} 1 & \\ & -\frac{1}{2} \end{pmatrix}}\Bigg|_{\lambda=\lambda_j} \\ &= (-2\det F_j^{\infty})^{-\frac{1}{2}} \times 2\mathrm{i}m_j F_j^{\infty}. \end{aligned}$$

Thus, we obtain

$$\Psi_\lambda \Psi^{-1} = -\frac{1}{4}\frac{1}{\lambda - \lambda_j}F_j^{\infty}\sigma_3(F_j^{\infty})^{-1} + \cdots = \frac{A_j}{\lambda - \lambda_j} + \cdots.$$

□

From theorem 5.1, we can obtain the following deformation equation:

Corollary 5.2. *The deformation equation of the monodromy-preserving deformation (5.1) is as follows. For* $j, k = 1, 2, \ldots, 2g+2$,

$$\mathrm{d}A_j = [\Theta_j, A_j],$$
$$\mathrm{d}F_j^{\infty} = \Theta_j A_j,$$
$$\Theta_j = \sum_{k \neq j} A_k \frac{\mathrm{d}\lambda_k - \mathrm{d}\lambda_j}{\lambda_k - \lambda_j} - \left[\Psi_{-1}^{\infty}, dB_{-1}\right] - d(\lambda_j B_{-1}),$$

where d *is the exterior differentiation with respect to the deformation parameters,* $t, \lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$.

Proof. See [4]. □

6. The τ-function for the Schlesinger system

In this section, we calculate the τ-function for the monodromy-preserving deformation (5.1). This section consists of three subsections. Subsection 5.1 is devoted to the Hamiltonian H_t. Subsections 5.2 and 5.3 are devoted to the Hamiltonian H_j ($j = 1, 2, \ldots, 2g+2$). In subsection 5.2, we quote Fay's identities and Rauch's variational formulas in order to compute the τ-function. In subsection 5.3, we calculate H_j and the τ-function.

6.1. The Hamiltonian at the irregular singular point $\lambda = \infty$

In this subsection, we prove proposition 6.1, where we compute ω_∞ and the Hamiltonian H_t.

Proposition 6.1.

$$\omega_\infty = \left(\frac{1}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \sum_{k=1}^{g} \frac{\partial}{\partial z_k} \{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))\} \frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^1}}(\infty^1) + \frac{t}{2}\left(\frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{\mathrm{d}x_{\infty^1}\mathrm{d}x_{\infty^2}}\right)\right)\mathrm{d}t$$
$$= H_t\, \mathrm{d}t.$$

Proof. We define

$$\Pi(P) = -\frac{t}{2}\left\{ -\int_{\lambda_1}^{P} W(P, \infty^1) + \int_{\lambda_1}^{P} W(P, \infty^2)\right\}, \tag{6.1}$$

and let $\hat{\Pi}(P)$ denote the regular part of $\Pi(P)$ around $\lambda = \infty^1$ which is given by

$$\hat{\Pi}(P) = -\frac{t}{2}\{\text{const} + \left(-\frac{1}{6} S(\infty^1) + \frac{W(\infty^1, \infty^2)}{\mathrm{d}x_{\infty^1}\, \mathrm{d}x_{\infty^2}}\right)\lambda^{-1} + \cdots\}. \tag{6.2}$$

Furthermore, we set

$$\hat{\varphi}(P) = \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\varphi) + \mathbf{v}(t))\Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\varphi)),$$
$$\hat{\psi}(P) = \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\psi) + \mathbf{v}(t))\Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\psi)).$$

Then, we get

$$\Psi(P) = \frac{\sqrt{\det \Phi(\infty)}}{\sqrt{\det \Phi(P)}}(G^\infty)^{-1}\Phi(P)$$
$$= \frac{\sqrt{\det \Phi(\infty)}}{\sqrt{\det \Phi(P)}}(G^\infty)^{-1}\begin{pmatrix} \hat{\varphi}(P)\exp(\hat{\Pi}(P)) & \hat{\varphi}(P^*)\exp(\hat{\Pi}(P^*)) \\ \hat{\psi}(P)\exp(\hat{\Pi}(P)) & \hat{\psi}(P^*)\exp(\hat{\Pi}(P^*)) \end{pmatrix}$$

$$\times \operatorname{diag}\left(\exp\left\{-\frac{t}{2}\lambda\right\}, \exp\left\{\frac{t}{2}\lambda\right\}\right)$$
$$:= \hat{\Psi}^{\infty}(\lambda)\exp T^{\infty}(\lambda).$$

From the definition of ω_∞, it follows that

$$\omega_\infty = -\mathrm{Res}_{\lambda=\infty}\mathrm{tr}\hat{\Psi}^{\infty}(\lambda)^{-1}\frac{\partial}{\partial\lambda}\hat{\Psi}^{\infty}(\lambda)\,\mathrm{d}T^{\infty}(\lambda). \tag{6.3}$$

In order to compute ω_∞, we set

$$A(\lambda) = (G^{\infty})^{-1}\begin{pmatrix} \hat{\varphi}(P)\exp\hat{\Pi}(P) & \hat{\varphi}(P^*)\exp\hat{\Pi}(P^*) \\ \hat{\psi}(P)\exp\hat{\Pi}(P) & \hat{\psi}(P^*)\exp\hat{\Pi}(P^*)\end{pmatrix}. \tag{6.4}$$

Therefore, we get

$$-\mathrm{tr}\hat{\Psi}^{\infty}(\lambda)^{-1}\frac{\partial}{\partial\lambda}\hat{\Psi}^{\infty}(\lambda)\,\mathrm{d}T^{\infty}(\lambda) = \frac{\mathrm{d}t}{2}\lambda\mathrm{tr}A^{-1}(\lambda)A'(\lambda)\begin{pmatrix}1 & \\ & -1\end{pmatrix} \tag{6.5}$$

and

$$\mathrm{tr}A^{-1}(\lambda)A'(\lambda)\begin{pmatrix}1 & \\ & -1\end{pmatrix}$$
$$= \frac{1}{\det\Phi(P)}\left[\det\begin{pmatrix}\{\hat{\varphi}(P)\exp\hat{\Pi}(P)\}' & \hat{\varphi}(P^*)\exp\hat{\Pi}(P^*) \\ \{\hat{\psi}(P)\exp\hat{\Pi}(P)\}' & \hat{\psi}(P^*)\exp\hat{\Pi}(P^*)\end{pmatrix}\right.$$
$$\left. -\det\begin{pmatrix}\hat{\varphi}(P)\exp\hat{\Pi}(P) & \{\hat{\varphi}(P^*)\exp\hat{\Pi}(P^*)\}' \\ \hat{\psi}(P)\exp\hat{\Pi}(P) & \{\hat{\psi}(P^*)\exp\hat{\Pi}(P^*)\}'\end{pmatrix}\right], \tag{6.6}$$

where $'$ means the differentiation with respect to the variable λ.

We have normalized the matrix function $\Psi(\lambda)$ around $\lambda = \infty$ in lemma 4.3 and have proved that the monodromy data of $\Psi(\lambda)$ are independent of P_φ, P_ψ in theorem 4.4 and its corollary. Therefore, we can choose the parameters, P_φ, P_ψ at our disposal to simplify the calculation.

Firstly, we multiply both the numerators and the denominators of (6.6) by $\frac{1}{\lambda_\psi - \lambda_\varphi}$. Then, we take the limit $P_\psi \to P_\varphi$ and get

$$\hat{\psi}(P) = \frac{\partial\hat{\varphi}(P)}{\partial\lambda_\varphi}. \tag{6.7}$$

Next, we multiply both the numerators and the denominators of (6.6) by $\frac{1}{\lambda_\varphi - \lambda}$. Then, we take the limit $P_\varphi \to P$ and obtain

$$\mathrm{tr}A^{-1}(\lambda)A'(\lambda)\begin{pmatrix}1 & \\ & -1\end{pmatrix} = 2\frac{1}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\lim_{P_\varphi\to P}\frac{\partial}{\partial\lambda_\varphi}\Theta[\mathbf{p},\mathbf{q}](-U(P)+U(P_\varphi)+\mathbf{v}(t))$$
$$+2\frac{\partial}{\partial\lambda}\{\hat{\Pi}(P)\}. \tag{6.8}$$

From the definition of ω_∞, it follows that

$$\omega_\infty = \frac{\mathrm{d}t}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\sum_{k=1}^{g}\frac{\partial}{\partial z_k}\{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))\}\frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^1}}(\infty^1)$$
$$+\frac{t}{2}\mathrm{d}t\left(\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}}\right).$$

□

6.2. Fay's identities and Rauch's variational formulas

In this subsection, we quote Fay's identities and Rauch's variational formulas in order to determine the Hamiltonians H_j $(j = 1, 2, \ldots, 2g + 2)$. Lemmas 6.2 and 6.3 are devoted to Fay's identities and Rauch's variational formulas, respectively.

Lemma 6.2. *(1) For $P, Q \in \mathcal{L}$,*

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T]^2(U(P) - U(Q))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]^2(0)E^2(P, Q)} = W(P, Q) + \sum_{k,l=1}^{g} \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0)\, \mathrm{d}U_k(P)\, \mathrm{d}U_l(Q).$$

(2) For $P, Q \in \mathcal{L}$,

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](2(U(P) - U(Q)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](0)E^4(P, Q)} - \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T]^4(U(P) - U(Q))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]^4(0)E^4(P, Q)}$$
$$= \frac{1}{2} \sum_{k,l,m,n=1}^{g} \frac{\partial^4}{\partial z_k \partial z_l \partial z_m \partial z_n} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0)\, \mathrm{d}U_k(P)\, \mathrm{d}U_l(P)\, \mathrm{d}U_m(Q)\, \mathrm{d}U_n(Q).$$

(3) For $P, Q \in \mathcal{L}$,

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](0)\Theta[\mathbf{p}, \mathbf{q}]\,(2\,(U(P) - U(Q)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]\,(U(P) - (Q))^2\, E(P, Q)^2} = W(P, Q)$$
$$+ \sum_{k,l=1}^{g} \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T]\,(U(P) - U(Q))\, \mathrm{d}U_k(P)\, \mathrm{d}U_l(Q)$$
$$= \mathrm{d}x_P\, \mathrm{d}x_Q \log \frac{E(P, Q)}{\Theta[\mathbf{p}^T, \mathbf{q}^T]\,(U(P) - U(Q))},$$

where x_P, x_Q are local coordinates of P, Q, respectively. Especially, since $\mathcal{L}$ is hyperelliptic, it follows that

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](0)\Theta[\mathbf{p}, \mathbf{q}]\,(2\,(U(P) - U(Q)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]\,(U(P) - (Q))^2\, E(P, Q)^2} = \mathrm{d}x_P\, \mathrm{d}x_Q \log \frac{1}{m_T(P, Q)}.$$

Proof. For (1), (2) and (3), see p 26, p 28 and p 29 in Fay's book [2], respectively. □

Rauch [9] described the dependence of $\mathrm{d}U_k$ $(k = 1, 2, \ldots, g)$ and $\mathbf{B}_{kl}$ $(k, l = 1, 2, \ldots, g)$ on the moduli of the Riemann surfaces. The moduli space of hyperelliptic curves can be parameterized by the positions of the branch points λ_j $(j = 1, 2, \ldots, 2g + 2)$. Korotkin [7] proved the variational formulas of the following useful form.

Lemma 6.3.

(1) For $P \in \mathcal{L}$,

$$\frac{\partial}{\partial \lambda_j} \left\{ \frac{\mathrm{d}U_k}{\mathrm{d}x_P}(P) \right\} = \frac{1}{2} \frac{W(P, \lambda_j)}{\mathrm{d}x_P\, dx_j} \frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j) \qquad (k = 1, 2, \ldots, g), \tag{6.9}$$

where x_P is a local coordinate of P and $x_j = \sqrt{\lambda - \lambda_j}$, which is a local coordinate of the branch point λ_j for any $j = 1, 2, \ldots, 2g + 2$.

(2) For the branch points λ_j $(j = 1, 2, \ldots, 2g + 2)$,

$$\frac{\partial \mathbf{B}_{kl}}{\partial \lambda_j} = \pi \mathrm{i} \frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j) \frac{\mathrm{d}U_l}{\mathrm{d}x_j}(\lambda_j) \qquad (k, l = 1, 2, \ldots, g), \tag{6.10}$$

where $x_j = \sqrt{\lambda - \lambda_j}$, which is a local coordinate of λ_j.

6.3. The τ-function

In this subsection, we compute ω_{λ_j} $(j = 1, 2, \ldots, 2g + 2)$ and calculate the τ-function.

Proposition 6.4. *For* $j = 1, 2, \ldots, 2g + 2$,

$$H_j = \frac{\partial}{\partial\lambda_j}\log\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t)|\mathbf{B}) - \frac{1}{2}\frac{\partial}{\partial\lambda_j}\log\det\mathcal{A} - \frac{1}{8}\frac{\partial}{\partial\lambda_j}\log\prod_{k<l}(\lambda_k-\lambda_l) + \frac{\partial}{\partial\lambda_j}\left\{\frac{t^2}{4}\left(\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}}\right)\right\}.$$

Proof. By direct calculation, we obtain

$$\frac{1}{2}\mathrm{tr}\left(\Psi'(\lambda)\Psi^{-1}(\lambda)\right)^2 = \frac{1}{4}\left(\frac{(\det\Phi)_\lambda}{\det\Phi}\right)^2 - \frac{\det(\Phi_\lambda)}{\det\Phi}.$$

We calculate ω_{λ_j} in the same way as ω_∞ in proposition 6.1.

We multiply both the numerators and the denominators of $\frac{(\det\Phi)_\lambda}{\det\Phi}, \frac{\det(\Phi_\lambda)}{\det\Phi}$ by $\frac{1}{\lambda_\varphi-\lambda_\psi}$. Then, we take the limit $P_\psi \to P_\varphi$ and get

$$\psi(P) = \frac{\partial\varphi(P)}{\partial\lambda_\varphi}. \tag{6.11}$$

Furthermore, we multiply both the numerators and the denominators of $\frac{(\det\Phi)_\lambda}{\det\Phi}, \frac{\det(\Phi_\lambda)}{\det\Phi}$ by $\frac{1}{\lambda_\varphi-\lambda}$ and take the limit $P_\varphi \to P$. Then, we obtain

$$\frac{(\det\Phi)_\lambda}{\det\Phi} = 2\frac{\partial}{\partial\lambda}\log\Theta[\mathbf{p}^S,\mathbf{q}^S](-2U(P)),$$

$$\begin{aligned}\frac{\det(\Phi_\lambda)}{\det\Phi} &= \frac{1}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\lim_{P_\varphi\to P}\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\Theta[\mathbf{p},\mathbf{q}](-U(P)+U(P_\varphi)+\mathbf{v}(t))\\ &+ \frac{2}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\lim_{P_\varphi\to P}\frac{\partial}{\partial\lambda}\Theta[\mathbf{p},\mathbf{q}](-U(P)+U(P_\varphi)+\mathbf{v}(t))\frac{\partial}{\partial\lambda}\Pi(P)\\ &+ \frac{1}{\Theta[\mathbf{p}^S,\mathbf{q}^S](-2U(P))}\lim_{P_\varphi\to P}\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\Theta[\mathbf{p}^S,\mathbf{q}^S](-U(P)-U(P_\varphi))\\ &- \left(\frac{\partial}{\partial\lambda}\Pi(P)\right)^2,\end{aligned}$$

which implies that

$$\frac{1}{2}\mathrm{tr}(\Psi'(\lambda)\Psi^{-1}(\lambda))^2 = -\left.\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\log\Theta[\mathbf{p}^S,\mathbf{q}^S](-U(P)-U(P_\varphi))\right|_{P_\varphi=P} \tag{6.12}$$

$$-\left.\frac{1}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\Theta[\mathbf{p},\mathbf{q}](-U(P)+U(P_\varphi)+\mathbf{v}(t))\right|_{P_\varphi=P} \tag{6.13}$$

$$-\left.\frac{2}{\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\frac{\partial}{\partial\lambda}\Theta[\mathbf{p},\mathbf{q}](-U(P)+U(P_\varphi)+\mathbf{v}(t))\frac{\partial}{\partial\lambda}\Pi(P)\right|_{P_\varphi=P} \tag{6.14}$$

$$+\left(\frac{\partial}{\partial\lambda}\Pi(P)\right)^2. \tag{6.15}$$

Firstly, we calculate the residue of (5.12) at $\lambda = \lambda_j$. For the purpose, we set

$$P = \lambda_j, \qquad x_P = x_j := \sqrt{\lambda-\lambda_j}$$

in (4.7) of lemma 4.2. Then, we have

$$\mathrm{Res}_{\lambda=\lambda_j}\left\{-\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\log\Theta[\mathbf{p}^S,\mathbf{q}^S](-U(P)-U(P_\varphi))\Big|_{P_\varphi=P}\right\}$$
$$=\frac{1}{8}\sum_{k\neq j}\frac{n_j n_k}{\lambda_j-\lambda_k}-\frac{1}{4\Theta[\mathbf{p}^T,\mathbf{q}^T](0|\mathbf{B})}\sum_{k,l=1}^{g}\frac{\partial^2\Theta[\mathbf{p}^T,\mathbf{q}^T]}{\partial z_k\partial z_l}(0|\mathbf{B})\frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j)\frac{\mathrm{d}U_l}{\mathrm{d}x_j}(\lambda_j),$$

where $n_k = 1$ for $\lambda_k \in T$ and $n_k = -1$ for $\lambda_k \notin T$. For the calculation, we use (6.10) in lemma 6.3 and the heat equation

$$\frac{\partial^2\Theta[\mathbf{p},\mathbf{q}](\mathbf{z}|\mathbf{B})}{\partial z_k\partial z_l}=4\pi\mathrm{i}\frac{\partial\Theta[\mathbf{p},\mathbf{q}](\mathbf{z}|\mathbf{B})}{\partial\mathbf{B}_{kl}}. \tag{6.16}$$

Then, we get

$$\mathrm{Res}_{\lambda=\lambda_j}\left\{-\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\log\Theta[\mathbf{p}^S,\mathbf{q}^S](-U(P)-U(P_\varphi))\Big|_{P_\varphi=P}\right\}$$
$$=\frac{1}{8}\sum_{k\neq j}\frac{n_j n_k}{\lambda_j-\lambda_k}-\frac{\partial}{\partial\lambda_j}\log\Theta[\mathbf{p}^T,\mathbf{q}^T](0|\mathbf{B}).$$

By using the Thomae's formula

$$\Theta^4[\mathbf{p}^T,\mathbf{q}^T](0|\mathbf{B})=\pm\frac{(\det\mathcal{A})^2}{(2\pi i)^{2g}}\prod_{l<k,l,k=1}^{g+1}(\lambda_{i_l}-\lambda_{i_k})\prod_{l<k,l,k=1}^{g+1}(\lambda_{j_l}-\lambda_{j_k}),$$

we get

$$\mathrm{Res}_{\lambda=\lambda_j}\left\{-\frac{\partial^2}{\partial\lambda\partial\lambda_\varphi}\log\Theta[\mathbf{p}^S,\mathbf{q}^S](-U(P)-U(P_\varphi))\Big|_{P_\varphi=P}\right\}$$
$$=-\frac{1}{2}\frac{\partial}{\partial\lambda_j}\log\det\mathcal{A}-\frac{1}{8}\frac{\partial}{\partial\lambda_j}\log\prod_{k<l}(\lambda_k-\lambda_l).$$

Secondly, we calculate the residue of the sum of (5.13) and (5.14) at $\lambda=\lambda_j$, which is

$$\frac{1}{4\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t)|\mathbf{B})}\sum_{k,l=1}^{g}\frac{\partial^2\Theta[\mathbf{p},\mathbf{q}]}{\partial z_k\partial z_l}(\mathbf{v}(t)|\mathbf{B})\frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j)\frac{\mathrm{d}U_l}{\mathrm{d}x_j}(\lambda_j)$$
$$+\frac{1}{2\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t))}\sum_{k=1}^{g}\frac{\partial\Theta[\mathbf{p},\mathbf{q}]}{\partial z_k}(\mathbf{v}(t))\times t\times\frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j)\frac{W(\infty^1,\lambda_j)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_j}. \tag{6.17}$$

From lemma 6.3, it follows that (6.17) is

$$\frac{\partial}{\partial\lambda_j}\log\Theta[\mathbf{p},\mathbf{q}](\mathbf{v}(t)|\mathbf{B}).$$

Lastly, we deal with the residue of (5.15) at $\lambda=\lambda_j$, which is $\frac{t^2}{4}\left(\frac{W(\lambda_j,\infty^1)}{\mathrm{d}x_j\,\mathrm{d}x_{\infty^1}}\right)^2$. We prove that if $\lambda_j\in T$,

$$\frac{\partial}{\partial\lambda_j}\left\{\frac{t^2}{4}\left(\frac{1}{6}S(\infty^1)-\frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}}\right)\right\}=\frac{t^2}{4}\left(\frac{W(\lambda_j,\infty^1)}{\mathrm{d}x_j\,\mathrm{d}x_{\infty^1}}\right)^2.$$

If $\lambda_j\notin T$, this formula can be proved in the same way.

From lemmas 4.2 and 6.2 (1), it follows that

$$\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}} = \frac{1}{16}\left(\sum_{k=1}^{g+1}\lambda_{i_k} - \sum_{k=1}^{g+1}\lambda_{j_k}\right)^2$$
$$- \frac{\Theta[\mathbf{p}^T,\mathbf{q}^T]^2(U(\infty^1)-U(\infty^2))}{\Theta[\mathbf{p}^T,\mathbf{q}^T]^2(0)\{E(\infty^1,\infty^2)\sqrt{\mathrm{d}x_{\infty^1}}\sqrt{\mathrm{d}x_{\infty^2}}\}^2}$$
$$-2\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\Theta[\mathbf{p}^T,\mathbf{q}^T](0|\mathbf{B})\frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^1}}(\infty^1)\frac{\mathrm{d}U_l}{\mathrm{d}x_{\infty^1}}(\infty^1)$$
$$= \frac{1}{8}\left(\sum_{k=1}^{g+1}\lambda_{i_k} - \sum_{k=1}^{g+1}\lambda_{j_k}\right)^2$$
$$-2\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\Theta[\mathbf{p}^T,\mathbf{q}^T](0|\mathbf{B})\frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^1}}(\infty^1)\frac{\mathrm{d}U_l}{\mathrm{d}x_{\infty^1}}(\infty^1),$$

where we have used the equation which is given by

$$\frac{\Theta[\mathbf{p}^T,\mathbf{q}^T](U(\infty^1)-U(\infty^2))}{\Theta[\mathbf{p}^T,\mathbf{q}^T](0)E(\infty^1,\infty^2)} = m_T(\infty^1,\infty^2) = \frac{\sqrt{-1}}{4}\left(\sum_{k=1}^{g+1}\lambda_{i_k} - \sum_{k=1}^{g+1}\lambda_{j_k}\right)\sqrt{\mathrm{d}x_{\infty^1}}\sqrt{\mathrm{d}x_{\infty^2}}.$$

Then, from lemma 6.3, it follows that

$$\frac{\partial}{\partial\lambda_j}\left\{\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}}\right\}$$
$$= \frac{1}{4}\left(\sum_{k=1}^{g+1}\lambda_{i_k} - \sum_{k=1}^{g+1}\lambda_{j_k}\right)$$
$$-\frac{1}{2}\sum_{k,l,m,n=1}^{g}\frac{\partial^4}{\partial z_k\partial z_l\partial z_m\partial z_n}\log\Theta[\mathbf{p}^T,\mathbf{q}^T](0)\frac{\mathrm{d}U_k}{\mathrm{d}x_{\infty^1}}(\infty^1)\frac{\mathrm{d}U_l}{\mathrm{d}x_{\infty^1}}(\infty^1)\frac{\mathrm{d}U_m}{\mathrm{d}x_j}(\lambda_j)\frac{\mathrm{d}U_n}{\mathrm{d}x_j}(\lambda_j)$$
$$-\left(\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\Theta[\mathbf{p}^T,\mathbf{q}^T](0)\frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j)\frac{\mathrm{d}U_l}{\mathrm{d}x_{\infty^1}}(\infty^1)\right)^2$$
$$-2\left(\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\Theta[\mathbf{p}^T,\mathbf{q}^T](0)\frac{\mathrm{d}U_k}{\mathrm{d}x_j}(\lambda_j)\frac{\mathrm{d}U_l}{\mathrm{d}x_{\infty^1}}(\infty^1)\right)\frac{W(\lambda_j,\infty^1)}{\mathrm{d}x_j\,\mathrm{d}x_{\infty^1}}.$$

Thus, from lemma 6.2 (1) and (2), we have

$$\frac{\partial}{\partial\lambda_j}\left\{\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}}\right\} = \frac{1}{4}\left(\sum_{k=1}^{g+1}\lambda_{i_k} - \sum_{k=1}^{g+1}\lambda_{j_k}\right)$$
$$-\frac{\Theta[\mathbf{p}^T,\mathbf{q}^T]\left(2\left(U(\infty^1)-U(\lambda_j)\right)\right)}{\Theta[\mathbf{p}^T,\mathbf{q}^T](0)\{E(\lambda_j,\infty^1)\sqrt{\mathrm{d}x_j}\sqrt{\mathrm{d}x_{\infty^1}}\}^4} + \left(\frac{W(\lambda_j,\infty^1)}{\mathrm{d}x_j\,\mathrm{d}x_{\infty^1}}\right)^2.$$

Therefore, from lemma 6.2 (3), we obtain

$$\frac{\partial}{\partial\lambda_j}\left\{\frac{1}{6}S(\infty^1) - \frac{W(\infty^1,\infty^2)}{\mathrm{d}x_{\infty^1}\,\mathrm{d}x_{\infty^2}}\right\} = \left(\frac{W(\lambda_j,\infty^1)}{\mathrm{d}x_j\,\mathrm{d}x_{\infty^1}}\right)^2,$$

which completes the proof of the proposition. □

Since we have calculated the Hamiltonians H_t and $H_1, H_2, \ldots, H_{2g+2}$, we can prove our main theorem.

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